Equations 3.1 through 3.15 are all that's necessary to solve any kinemetric and dynamics problem in nonorthogonal coordinates. These equations are also very useful in solving inertial guidance problems using nonorthogonal sensors.

4.0 Specialization to Orthogonal Coordinate Systems

The use of orthonormal coordinate bases has many practical advantages when the problem permits such use. It will be seen that this case permits key simplifications to be made to the two fundamental matrix operations defined in the previous section since the reciprocal basis for a orthonormal set is itself. Hence all the equations derived previously can be used by noting that $c = 1$, $(\bar{e}_A) \equiv (\bar{e}_A{}^*)$.

Appendix A: Rotational Problems in Nonorthogonal Basis

Lets assume that we have a rotating satellite in space, where the rate gyros, and thrusters (moment generators) are located in two different nonorthogonal bases A and B, respectively. Further, lets assume that the inertial tensor $\bar{\bar{I}}$ is given in yet another basis say C. It is desired to find the dynamic equation (differential equation) governing the time history of the gyro outputs.

$$\bar{\bar{I}} = (\bar{e}_C)^T[I_C](\bar{e}_C) \quad \text{(inertial tensor)} \tag{A1}$$

$$\bar{M} = (\bar{e}_B)^T(M_B) \quad \text{(torque equation)} \tag{A2}$$

$$\bar{\omega} = (\omega_A)^T(\bar{e}_A) \quad \text{(angular velocity)} \tag{A3}$$

Now, we can proceed to solve the dynamic problem

$$\begin{aligned}\bar{H} &= \bar{\bar{I}}\cdot\bar{\omega} \\ &= (\bar{e}_C)^T[I_C](\bar{e}_C)\cdot(\bar{e}_A{}^*)^T(\omega_A) \\ &= (\bar{e}_A)^T[T_{CA}]^T[I_C][T_{CA}](\omega_A) \\ &\triangleq (\bar{e}_A)^T(h_A{}^*)\end{aligned} \tag{A4}$$

Applying the 2nd law of Newton, we have

$$\begin{aligned}d\bar{H}/dt &= (\bar{e}_A)^T[(\dot{h}_A{}^*) + c_A[T_{A*A}][\tilde{\omega}_A{}^*]_A(h_A{}^*)] \\ &= (e_A)^T[(\dot{h}_A{}^*) + c_A{}^*[\tilde{\omega}_A]_A[T_{AA}{}^*](h_A{}^*)] \\ &= \bar{M} = (\bar{e}_B)^T(M_B)\end{aligned} \tag{A5}$$

Finally, we have the desired nonlinear differential equation

$$(\dot{h}_A{}^*) + c_A{}^*[\tilde{\omega}_A]_A[T_{AA*}](h_A{}^*) = [T_{BA}]^T(M_B) \tag{A6}$$

where

$$(h_A{}^*) \triangleq [T_{CA}]^T[I_C][T_{CA}](\omega_A)$$

Admittedly, Eq. (A6) is still rather complicated, but it is no more difficult to solve than any other coupled nonlinear 1st order equations. The purpose of this problem is to illustrate how a seemingly very difficult problem can be attacked systematically using the techniques outlined in this paper.

Appendix B: Some Basic Relationships in Nonorthogonal Coordinate Systems

Relationship between c and c*

$c \triangleq (\bar{e}_3)\cdot(\bar{e}_1 \times \bar{e}_2) \triangleq \mathrm{Det}[E]$

= volume of parallelipiped formed by the vectors $\bar{e}_1,\bar{e}_2,\bar{e}_3$, (B1)

Note that the elements E_{11}, E_{12}, E_{13} are the projections of $\bar{e}_1$ onto any arbitrary orthogonal coordinate system.

$$c^* \triangleq (\bar{e}_3{}^*)\cdot(e_1{}^* \times \bar{e}_2{}^*) \triangleq \mathrm{Det}[E^*]$$

Note that

$$[EE^*] = [U]$$

$$\therefore \mathrm{Det}[EE^*] = \mathrm{Det}[E]\mathrm{Det}[E^*] = cc^* = 1$$

Relationship between $[\tilde{w}]$ and $[\tilde{w}^*]$

Since

$$(\bar{e}_A)\cdot(\bar{e}_A{}^*)^T = [U]$$

Differentiating and rearranging, we have

$$[\tilde{\omega}_A{}^*]_A = \frac{c^*}{c} T_{AA*}[\tilde{\omega}_A]_A T_{AA*} \tag{B2}$$

Relative Angular Velocities

The relative angular velocity between frame A and B expressed in frame A is

$$(\bar{\omega}_{AB})_A \triangleq (\bar{\omega}_A)_A - (\bar{\omega}_B)_A$$

$$\therefore [\tilde{\omega}_{AB}]_A = [\tilde{\omega}_A{}^*]_A - [\tilde{\omega}_B{}^*]_A \tag{B3}$$

where

$$\begin{aligned}[\tilde{\omega}_B{}^*]_A{}^T &= (c_B/c_A)[T_{AB}][\tilde{\omega}_B{}^*]_B{}^T[T_{B*B}][T_{BA}][T_{AA*}] \\ &= (c_B{}^*/c_A)[T_{AB}][T_{BB*}][\tilde{\omega}_B]_B{}^T[T_{BA}][T_{AA*}]\end{aligned} \tag{B4}$$

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Reissner's Plate Equations in Polar Coordinates

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In a recent paper by Lehnhoff and Miller[1] the influence of transverse shear on circular plates has been investigated using a method suggested by Goodier. The resulting plate equations in polar coordinates have been solved using Fourier Analysis. In this Note, the effect of shear deformations on circular plates is studied using Reissner's theory. The plate equations thus obtained in polar coordinates have been reduced in general, to the solution of equations governing the lateral deflection and a stress function. Bending moments and shear forces are expressed as functions of lateral deflection and the stress function. As an example, complete solutions are presented for a circular plate subjected to arbitrary lateral load.

Equilibrium equations governing the bending of a plate in polar coordinates are

$$\partial M_r/\partial r + (1/r)\partial M_{r\theta}/\partial\theta + (M_r - M_\theta)/r - Q_r = 0 \tag{1}$$

$$\partial M_{r\theta}/\partial r + (1/r)\partial M_\theta/\partial\theta + 2M_{r\theta}/r - Q_\theta = 0 \tag{2}$$

$$\partial Q_r/\partial r + Q_r/r + (1/r)\partial Q_\theta/\partial\theta + q = 0 \tag{3}$$

where $q(r,\theta)$ is the lateral load acting on the plate. Reissner's

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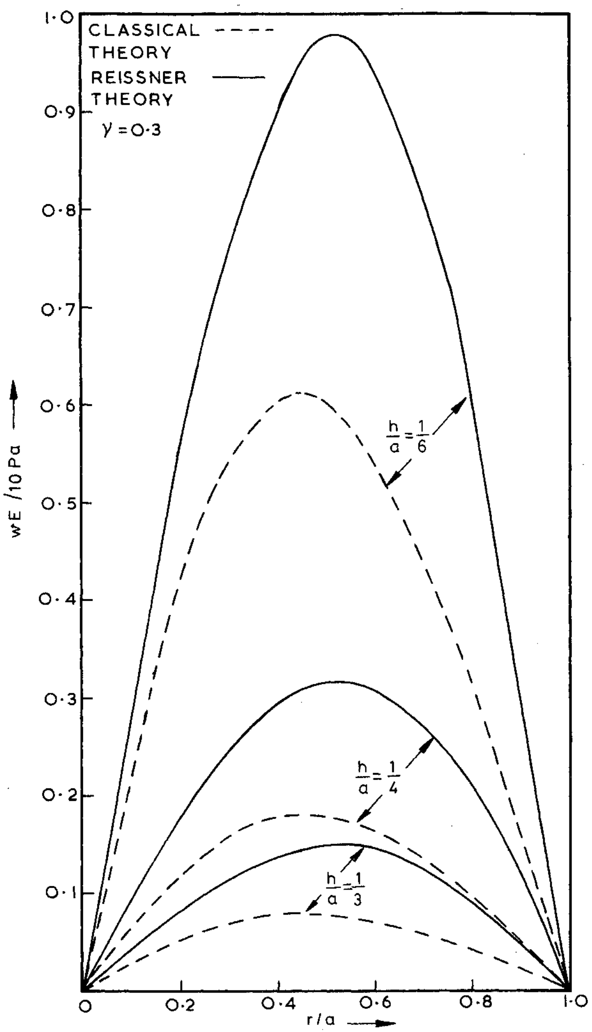

Fig. 1 Variation of w.

moment-curvature relations[2] are

$$M_r - \frac{h^2}{5}\frac{\partial Q_{r\theta}}{\partial r} + \frac{\nu h^2}{10(1-\nu)}\, q = -D\left\{\frac{\partial^2 w}{\partial r^2} + \nu\left(\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right)\right\} \quad (4)$$

$$M_\theta - \frac{h^2}{5}\left(\frac{1}{r}\frac{\partial Q_\theta}{\partial \theta} + \frac{Q_r}{r}\right) + \frac{\nu h^2}{10(1-\nu)}\, q = -D\left\{\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + \nu\frac{\partial^2 w}{\partial r^2}\right\} \quad (5)$$

$$M_{r\theta} - \frac{h^2}{10}\left(\frac{1}{r}\frac{\partial Q_r}{\partial \theta} + \frac{\partial Q_\theta}{\partial r} - \frac{Q_\theta}{r}\right) = -D(1+\nu)\left(\frac{1}{r}\frac{\partial^2 w}{\partial_r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right) \quad (6)$$

where D is the flexural rigidity and ν is Poisson's ratio.

Substituting Eqs. (4–6) into Eqs. (1) and (2), force-displacement relations are obtained as

$$Q_r - \frac{h^2}{10}\left(\nabla^2 Q_r - \frac{Q_r}{r^2} - \frac{2}{r^2}\frac{\partial Q_\theta}{\partial \theta}\right) + \frac{h^2}{10(1-\nu)}\frac{\partial q}{\partial r} = -D\frac{\partial}{\partial r}(\nabla^2 w) \quad (7)$$

$$Q_\theta - \frac{h^2}{10}\left(\nabla^2 Q_\theta - \frac{Q_\theta}{r^2} + \frac{2}{r^2}\frac{\partial Q_r}{\partial \theta}\right) + \frac{h^2}{10(1-\nu)}\frac{1}{r}\frac{\partial q}{\partial \theta} = -D\frac{1}{r}\frac{\partial}{\partial \theta}(\nabla^2 w) \quad (8)$$

Substitution of Eqs. (7) and (8) in Eq. (3) will result in an equation governing the lateral deflection

$$D\nabla^2\nabla^2 w = q - (h^2/10)[(2-\nu)/(1-\nu)]\nabla^2 q \quad (9)$$

Eqs. (7–9) have to be solved with proper boundary conditions chosen from the following boundary equations:

along circumferential direction

$$\begin{aligned} &M_r \cdot \{(12/5)[(1+\nu/Eh]Q_r - \partial w/\partial r\} = 0 \\ &M_{r\theta} \cdot \{(12/5)[(1+\nu)/Eh]Q_\theta - \partial w/\partial \theta\} = 0,\ Q_r \cdot w = 0 \end{aligned} \quad (10)$$

along radial direction

$$\begin{aligned} &M_\theta \cdot \{(12/5)[(1+\nu/Eh]Q_\theta - (1/r)\partial w/\partial \theta\} = 0 \\ &M_{r\theta} \cdot \{(12/5)[(1+\nu)/Eh]Q_r - \partial w/\partial r\} = 0,\ Q_\theta \cdot w = 0 \end{aligned} \quad (11)$$

Particular integrals of Eqs. (7) and (8), [if the solution of Eq. (9) is known], are given by

$$Q_{rp} = -D(\partial/\partial r)(\nabla^2 w) - (h^2/10)[(2-\nu)/(1-\nu)]\partial q/\partial r \quad (12)$$

$$Q_{\theta p} = -D(1/r)(\partial/\partial \theta)(\nabla^2 w) - (h^2/10r)[(2-\nu)/(1-\nu)]\partial q/\partial \theta \quad (12a)$$

Now, the homogeneous solutions of Eqs. (7) and (8) are obtained by a stress function $F(r,\theta)$ such that

$$Q_{rh} = (1/r)\partial F/\partial \theta; \quad Q_{\theta h} = -\partial F/\partial r \quad (13)$$

Solutions given by Eq. (13) will satisfy the homogeneous part of the equilibrium Eq. (3). Governing equation for F is obtained by substituting Eq. (13) into the homogeneous part of Eqs. (7) and (8) as

$$\nabla^2 F - (10/h^2)F = 0 \quad (14)$$

The problem reduces to solving Eqs. (9) and (14). The bending and twisting moments take the following form in terms of F and w

$$M_r = \frac{h^2}{5}\left(\frac{1}{r}\frac{\partial^2 F}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial F}{\partial \theta}\right) - D\left[\nu\nabla^2 w + (1-\nu)\frac{\partial^2 w}{\partial r^2} - \frac{h^2}{5}\left\{\frac{1}{r}\frac{\partial}{\partial r}(\nabla^2 w) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}(\nabla^2 w)\right\}\right] - \frac{h^2}{10}\left(\frac{2-\nu}{1-\nu}\right)\left[q - \frac{h^2}{5}\left(\frac{1}{r}\frac{\partial q}{\partial r} + \frac{1}{r^2}\frac{\partial^2 q}{\partial \theta^2}\right)\right] \quad (15)$$

$$M_\theta = -\frac{h^2}{\cdot 5}\left(\frac{1}{r}\frac{\partial^2 F}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial F}{\partial \theta}\right) - D\left[\nabla^2 w - (1-\nu)\times\frac{\partial^2 w}{\partial r^2} - \frac{h^2}{5}\frac{\partial^2}{\partial r^2}(\nabla^2 w)\right] - \frac{h^2}{10}\left(\frac{2-\nu}{1-\nu}\right)\left[q - \frac{h^2}{5}\frac{\partial^2 q}{\partial r^2}\right]\ldots \quad (16)$$

$$M_{r\theta} = \frac{h^2}{10}\left(\frac{1}{r^2}\frac{\partial^2 F}{\partial \theta^2} - \frac{\partial^2 F}{\partial r^2} + \frac{1}{r}\frac{\partial F}{\partial r}\right) - D\left[(1-\nu)\times\left(\frac{1}{r}\frac{\partial^2 w}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial w}{\partial \theta}\right) + \frac{h^2}{5}\left\{\frac{1}{r}\frac{\partial^2}{\partial r \partial \theta}(\nabla^2 w) - \frac{1}{r^2}\frac{\partial}{\partial \theta}(\nabla^2 w)\right\}\right] - \frac{h^2}{10}\left(\frac{2-\nu}{1-\nu}\right)\left[\frac{h^2}{5}\left(\frac{1}{r}\frac{\partial^2 q}{\partial r \partial \theta} - \frac{1}{r^2}\frac{\partial q}{\partial \theta}\right)\right]\ldots \quad (17)$$

Considering a circular plate as an example, the lateral load on the plate can be represented as Fourier expansion in θ

$$q(r,\theta) = \sum_{m=0}^{\infty}[q_m(r)\cos m\theta + \bar{q}_m(r)\sin m\theta] \quad (18)$$

Solutions of Eq. (9) can be written in the form

$$w(r,\theta) = \sum_{m=0}^{\infty}[R_m(r)\cos m\theta + \bar{R}_m(r)\sin m\theta] + w_p(r,\theta) \quad (19)$$

where $w_p(r,\theta)$ is the particular integral of Eq. (9) and $R_m(r)$

is given by

$$R_0 = A_0 + B_0r^2 + C_0 \log r + D_0r^2 \log r \tag{20a}$$

$$R_1 = A_1r + B_1r^3 + C_1r^{-1} + D_1r \log r \tag{20b}$$

$$R_m = A_mr^m + B_mr^{-m} + C_mr^{m+2} + D_mr^{-m+2} \tag{20c}$$

$\bar{R}_m(r)$ is the same form as in Eqs. (20) except for different constants.

The solution of Eq. (14) will be in the form

$$F(r,\theta) = \sum_{m=0}^{\infty} [F_m(r) \sin m\theta + \bar{F}_m(r) \cos m\theta] \tag{21}$$

where $F_m(r)$ is given by

$$F_m(r) = \bar{C}_mI_m(Gr/h) + \bar{D}_mK_m(Gr/h) \tag{22}$$

where I_m and K_m are modified Bessel functions and

$$G = (10)^{1/2}$$

$\bar{F}_m(r)$ is of the same form except for different constants. The six set of constants in Eqs. (20) and (22) are obtained by satisfying the appropriate boundary conditions along circumferential directions.

As a particular example, consider a circular plate, simply supported along $r = a$ with a lateral load in the form

$$q = Pr \cos\theta/a \tag{23}$$

Boundary conditions in this case are

$$w = 0; \quad M_r = 0; \quad (12/5)[(1 + \nu)/Eh]Q_\theta - (1/r)\partial w/\partial\theta = 0 \tag{24}$$

along $r = a$ and the solutions should be finite at $r = 0$. In order to keep the solutions finite at $r = 0$, the constants C_1 and D_1 in the expression for $R_1(r)$ in Eq. (20) and the constant $\bar{D}_1$ in the expression for $F_1(r)$ in Eq. (22) are taken to be zero. The remaining three constants in the expressions R_1 and F_1 are determined by the conditions given in Eq. (24). Lateral deflection and stress functions are given by

$$w = \frac{Pa^4}{192D(3+\nu)}\left[\frac{r}{a}\left(1 - \frac{r^2}{a^2}\right)\left\{(7+\nu) - (3+\nu)\frac{r^2}{a^2} + \frac{h^2}{a^2} f\left(G\frac{a}{h}\right)\right\}\right] \cos\theta \ldots \tag{25a}$$

$$F = -\frac{Pa^2}{24}\left\{g\left(G\frac{a}{h}\right) I_1\left(G\frac{r}{h}\right)\right\} \sin\theta \tag{25b}$$

where

$$g\left(G\frac{a}{h}\right) = \frac{(1-\nu) - \dfrac{2\nu}{5}\dfrac{h^2}{a^2}}{(3+\nu)\left(G\dfrac{a}{h}I_0 - I_1\right) - \dfrac{4}{5}\left(G\dfrac{a}{h}I_0 - 2I_1\right)\dfrac{h^2}{a^2}} \tag{26a}$$

$$f(Ga/g) = \tfrac{4}{5}[6[(3-\nu)/(1-\nu)] + (Ga/hI_0 - 2I_1)g] \tag{26b}$$

In Eqs. (26) the terms I_0 and I_1 are evaluated with the argument Ga/h, that is

$$I_0 = I_0(Ga/h) \text{ and } I_1 = I_1(Ga/h) \tag{27}$$

Bending moments and shear forces are given by

$$M_r = \frac{Pa^2}{48}\left((5+\nu)\frac{r}{a}\left(1 - \frac{r^2}{a^2}\right) + \frac{h^2}{a^2}\left\{\left(\frac{1}{2}f - \frac{12}{5}\frac{3-\nu}{1-\nu}\right)\frac{r}{a} - \frac{2}{5}g\left[\frac{a}{r}\frac{Ga}{h}I_0\left(\frac{Gr}{h}\right) - 2\frac{a^2}{r^2}I_1\left(G\frac{r}{h}\right)\right]\right\}\right)\cos\theta$$

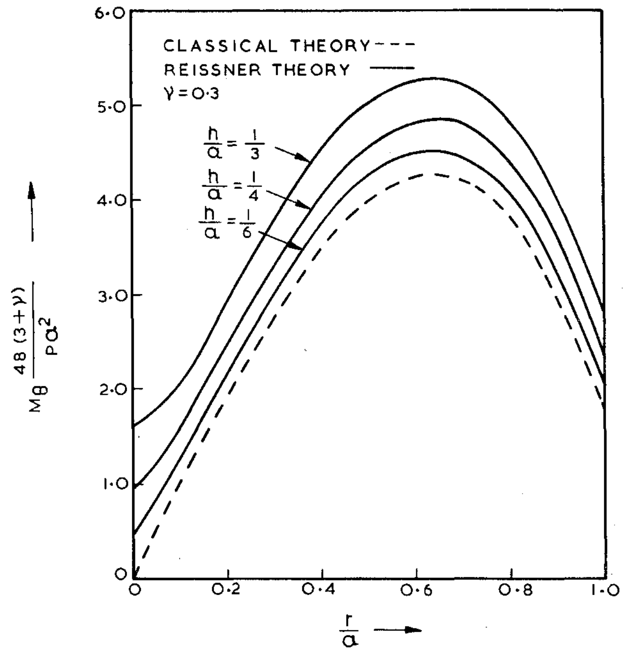

Fig. 2 Variation of M_θ.

$$M_\theta = \frac{Pa^2}{48(3+\nu)}\left(\frac{r}{a}\left\{(1+3\nu)(5+\nu) - (1+5\nu)(3+\nu)\frac{r^2}{a^2} - \frac{h^2}{a^2}\frac{7-5\nu}{20(1-\nu)}\right\} + \frac{h^2}{a^2}\left\{\frac{1+3\nu}{2}f + \frac{2}{5}g\left[\frac{a}{r}G\frac{a}{h}I_0\left(G\frac{r}{h}\right) - 2\frac{a^2}{r^2}I_1\left(G\frac{r}{h}\right)\right]\right\}\right)\cos\theta$$

$$Q_r = \frac{Pa}{24(3+\nu)}\left[2(5+\nu) - 9(3+\nu)\frac{r^2}{a^2} + \frac{h^2}{a^2}\left\{f - \frac{12}{5}\frac{(2-\nu)(3+\nu)}{(1-\nu)}\right\} - (3+\nu)g\frac{a}{r}I_1\left(G\frac{r}{h}\right)\right]\cos\theta$$

$$Q_\theta = -\frac{Pa}{24(3+\nu)}\left[2(5+\nu) - 3(3+\nu)\frac{r^2}{a^2} + \frac{h^2}{a^2}\left\{f - \frac{12}{5}\frac{(2-\nu)(3+\nu)}{(1-\nu)}\right\} - (3+\nu)\left\{G\frac{a}{h}I_0\left(G\frac{r}{h}\right) - \frac{a}{r}I_1\left(G\frac{h}{r}\right)\right\}g\right]\sin\theta \tag{28}$$

where the quantities "f" and "g" are given by Eqs. (26).

In Eqs. (28) and (25a) if the terms multiplied by h/a are omitted, the expressions due to classical plate theory[3] can be obtained.

Results are computed for values of $h/a = \frac{1}{3}, \frac{1}{4}$, and $\frac{1}{6}$ and the computed values of w and M_θ (obtained from the present work) have been compared with the solutions from the classical theory in Figs. 1 and 2. It is observed in this example, that the values of M_r, Q_r, and Q_θ are not much affected by the transverse shear. This may be due to the type of boundary conditions[4] considered in this example.

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